Dirichlet L-Series.

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Dirichlet L-Series

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A Dirichlet L-series is a series of the form

$$L_k(s,\chi) \equiv \sum_{n=1}^{\infty} \chi_k(n) n^{-s},$$

where the number theoretic character $\chi_k(n)$ is an integer function wit
series are very important in additive number theory (they were used, fo
have a close connection with modular forms. Dirichlet L-series can be
power of $e^{2\pi i/k}$.

Dirichlet L-series is implemented in the Wolfram Language as Dirichlet
modulus k and index j.

The generalized Riemann hypothesis conjectures that neither the Riem
a zero with real part larger than 1/2.

The Dirichlet lambda function

$$\lambda(s) \equiv \sum_{n=0}^{\infty} \frac{1}{(2n+1)^s} = (1-2^{-s})\zeta(s),$$

Dirichlet beta function

$$L_{-4}(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s} = \beta(s)$$

and Riemann zeta function

$$L_{+1}(s) = \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

are all Dirichlet L-series (Borwein and Borwein 1987, p. 289).

Hecke (1936) found a remarkable connection between each modular fo

$$f(\tau) = c(0) + \sum_{n=1}^{\infty} c(n) e^{2\pi i n \tau}$$

and the Dirichlet L-series

$$\phi(s) = \sum_{n=1}^{\infty} \frac{c(n)}{n^s}$$

This Dirichlet series converges absolutely for $\sigma = \mathbb{R}[s] > k + 1$ (if f is
form. In particular, if the coefficients $c(n)$ satisfy the multiplicative pro

$$c(m) c(n) = \sum_{d|(m,n)} d^{2k-1} c\left(\frac{m n}{d^2}\right),$$

then the Dirichlet L-series will have a representation of the form

$$\phi(s) = \prod_p \frac{1}{1 - c(p) p^{-s} + p^{2k-1} p^{-2s}},$$

which is absolutely convergent with the Dirichlet series (Apostol 1997, p
integer, then $\phi(s)$ can be analytically continued beyond the line $\sigma = k$ s

1. If $c(0) = 0$, then $\phi(s)$ is an entire function of s,

2. If $c(0) \neq 0$, $\phi(s)$ is analytic for all s except a single simple pole at $s =$

$$\frac{(-1)^{k/2} c(0) (2\pi)^k}{\Gamma(k)},$$

where $\Gamma(k)$ is the gamma function, and

3. $\phi(s)$ satisfies

$$(2\pi)^{-s} \Gamma(s) \phi(s) = (-1)^{k/2} (2\pi)^{s-k} \Gamma(k-s) \phi(k-s)$$

(Apostol 1997, p. 137).

The number theoretic character χ_k is called primitive if the j-conducto
primitive L-series modulo k is then defined as one for which $\chi_k(n)$ is p
expressed in terms of primitive L-series.

Let $P = 1$ or $P = \prod_{i=1}^{t} p_i$, where p_i are distinct odd primes. Then ther
with real coefficients. The requirement of real coefficients restricts the
all k and n. The three type are then

1. If $k = P$ (e.g., $k = 1$, 3, 5, ...) or $k = 4P$ (e.g., $k = 4$, 12, 20, ...), there is

2. If $k = 8P$ (e.g., $k = 8$, 24, ...), there are two primitive L-series.

3. If $k = 2P$, $P p_i$, or $2^\alpha P$ where $\alpha > 3$ (e.g., $k = 2$, 6, 9, ...), there are n

(Zucker and Robertson 1976). All primitive L-series are algebraically in
to

$$\chi_k(k-1) = \pm 1.$$

Primitive L-series of these types are denoted $L_{\pm}$. For a primitive L-seri

$k = P$, then

$$L_k = \begin{cases} L_{-k} & \text{if } P \equiv 3 \pmod 4 \\ L_k & \text{if } P \equiv 1 \pmod 4). \end{cases}$$

If $k = 4P$, then

$$L_k = \begin{cases} L_{-k} & \text{if } P \equiv 1 \pmod 4 \\ L_k & \text{if } P \equiv 3 \pmod 4), \end{cases}$$

and if $k = 8P$, then there is a primitive function of each type (Zucker a

The first few primitive negative L-series are L_{-3}, L_{-4}, L_{-7}, L_{-8}, L_{-11}, L
L_{-40}, L_{-43}, L_{-47}, L_{-51}, L_{-52}, L_{-55}, L_{-56}, L_{-59}, L_{-67}, L_{-68}, L_{-71}, L_{-79}, L
A003657), corresponding to the negated discriminants of imaginary qua
series are L_{+1}, L_{+5}, L_{+8}, L_{+12}, L_{+13}, L_{+17}, L_{+21}, L_{+24}, L_{+28}, L_{+29}, L_{+33},
L_{+61}, L_{+65}, L_{+69}, L_{+73}, L_{+76}, L_{+77}, L_{+85}, L_{+88}, L_{+89}, L_{+92}, L_{+93}, L_{+97}, ..

The Kronecker symbol (d/n) is a real number theoretic character mod
real primitive number theoretic character mod d (Ayoub 1963). Theref

$$L_d(s) = \sum_{n=1}^{\infty} (d/n)\, n^{-s}$$

where (d/n) is the Kronecker symbol (Borwein and Borwein 1987, p. 2

For primitive values of d, the Kronecker symbols are periodic with peri
$|d| - 1$ sums, each of which can be expressed in terms of the polygamm

$$L_d(s) = \frac{1}{(-|d|)^s (s-1)!} \sum_{n=1}^{|d|-1} (d/n)\, \psi_{s-1}\left(\frac{n}{|d|}\right).$$

The functional equations for $L_{\pm}$ are

$$L_{-d}(s) = 2^s \pi^{s-1} d^{-s+1/2}\, \Gamma(1-s) \cos\left(\tfrac{1}{2} s \pi\right) L_{-d}(1-s)$$
$$L_{+d}(s) = 2^s \pi^{s-1} d^{-s+1/2}\, \Gamma(1-s) \sin\left(\tfrac{1}{2} s \pi\right) L_{+d}(1-s)$$

(Borwein and Borwein 1986, p. 303).

For m a positive integer

$$L_{+d}(-2m) = 0$$
$$L_{-d}(1-2m) = 0$$
$$L_{+d}(2m) = R\, k^{-1/2}\, \pi^{2m}$$
$$L_{-d}(2m-1) = R'\, k^{-1/2}\, \pi^{2m-1}$$
$$L_{+d}(1-2m) = \frac{(-1)^m (2m-1)!\, R}{(2k)^{2m-1}}$$
$$L_{-d}(-2k) = \frac{(-1)^m R' (2m)!}{(2k)^{2m}},$$

where R and R' are rational numbers. Nothing general appears to be k
although it is possible to express all $L_{\pm}(1)$ in terms of known transcend

$L_{+d}(1)$ can be expressed in terms of transcendentals by

$$L_d(1) = h(d)\,\kappa(d),$$

where $h(d)$ is the class number and $\kappa(d)$ is the Dirichlet structure cons

No general forms are known for $L_{-d}(2m)$ and $L_{+d}(2m-1)$ in terms c
several examples of special cases of $L_d(1)$. A number of primitive serie

$$\begin{aligned}
L_{-20}(1) &= \frac{\pi}{\sqrt{5}} \\
L_{-15}(1) &= \frac{2\pi}{\sqrt{15}} \\
L_{-11}(1) &= \frac{\pi}{\sqrt{11}} \\
L_{-8}(1) &= \frac{\pi}{2\sqrt{2}} \\
L_{-7}(1) &= \frac{\pi}{\sqrt{7}} \\
L_{-4}(1) &= \tfrac{1}{4}\pi \\
L_{-3}(1) &= \tfrac{1}{9}\pi\sqrt{3} \\
L_{+5}(1) &= \tfrac{2}{5}\sqrt{5}\,\ln\phi \\
L_{+8}(1) &= \frac{\ln\left(1+\sqrt{2}\right)}{\sqrt{2}} \\
L_{+12}(1) &= \frac{\ln\left(2+\sqrt{3}\right)}{\sqrt{3}} \\
L_{+13}(1) &= \frac{2}{\sqrt{13}}\ln\left(\frac{3+\sqrt{13}}{2}\right) \\
L_{+17}(1) &= \frac{2}{\sqrt{17}}\ln\left(4+\sqrt{17}\right) \\
L_{+21}(1) &= \frac{2}{\sqrt{21}}\ln\left(\frac{5+\sqrt{21}}{2}\right) \\
L_{+24}(1) &= \frac{\ln\left(5+2\sqrt{6}\right)}{\sqrt{6}},
\end{aligned}$$

and for $L_k(2)$ are given by

$$\begin{aligned}
L_{-8}(2) &= \tfrac{1}{64}\left[\psi_1\left(\tfrac{1}{8}\right)+\psi_1\left(\tfrac{3}{8}\right)-\psi_1\left(\tfrac{5}{8}\right)-\psi_1\left(\tfrac{7}{8}\right)\right] \\
L_{-7}(2) &= \tfrac{1}{49}\left[\psi_1\left(\tfrac{1}{7}\right)+\psi_1\left(\tfrac{2}{7}\right)-\psi_1\left(\tfrac{3}{7}\right)+\psi_1\left(\tfrac{4}{7}\right)\right. \\
L_{-4}(2) &= K \\
L_{-3}(2) &= \tfrac{1}{9}\left[\psi_1\left(\tfrac{1}{3}\right)-\psi_1\left(\tfrac{2}{3}\right)\right] \\
L_{+1}(2) &= \tfrac{1}{6}\pi^2 \\
L_{+5}(2) &= \tfrac{4}{125}\pi^2\sqrt{5} \\
L_{+8}(2) &= \tfrac{1}{16}\pi^2\sqrt{2} \\
L_{+12}(2) &= \tfrac{1}{18}\pi^2\sqrt{3}
\end{aligned}$$

$$L_{+13}(2) = \frac{4\pi^2}{13\sqrt{13}}$$

$$L_{+17}(2) = \frac{8\pi^2}{17\sqrt{17}}$$

$$L_{+21}(2) = \frac{8\pi^2}{21\sqrt{21}},$$

where K is Catalan's constant, $\psi_1(z)$ is the trigamma function, and Li_2

Bailey and Borwein (Bailey and Borwein 2005; Bailey *et al.* 2006a, pp. 5 a
2008; Coffey 2008) conjectured the relation actually in effect proved by
(M. Coffey, pers. comm., Mar. 30, 2009) that $L_{-7}(2)$ is also given by

$$I_7 = \frac{24}{7\sqrt{7}} \int_{\pi/3}^{\pi/2} \ln\left|\frac{\tan x + \sqrt{7}}{\tan x - \sqrt{7}}\right| dx$$

$$- \frac{4}{7\sqrt{7}} \left\{9 \ln 2 \cot^{-1}\sqrt{7} + \left(\pi - 6\cot^{-1}\sqrt{7}\right) \times \ln\right.$$

$$= \mathrm{Li}_2\left(\frac{\sqrt{7}-\sqrt{3}}{\sqrt{7}-i}\right) - \mathrm{Li}_2\left(\frac{\sqrt{7}-\sqrt{3}}{\sqrt{7}+i}\right) - \mathrm{Li}_2\left(\ldots\right.$$

$$= \frac{24}{7\sqrt{7}} \left\{\mathrm{Cl}_2(\theta_+) + \tfrac{1}{2}\left[\mathrm{Cl}_2(2\omega_+) - \mathrm{Cl}_2(2\omega_+ + 2\theta_+\ldots\right.\right.$$

$$= \frac{4}{7\sqrt{7}} \left[3\,\mathrm{Cl}_2(\theta_7) - 3\,\mathrm{Cl}_2(2\theta_7) + \mathrm{Cl}_2(3\theta_7)\right]$$

$$= 1.1519254705\ldots$$

where the latter expressions are due to Coffey (2008ab), with

$$\omega_+ = \tan^{-1}\left(\sqrt{7}\right) - \frac{2\pi}{3}$$

$$\omega_- = -\omega_+$$

$$= \tan^{-1}\left(\frac{2\sqrt{3}-\sqrt{7}}{5}\right)$$

$$\theta_+ = \tan^{-1}\left(\tfrac{1}{3}\sqrt{7}\right)$$

$$\theta_7 = 2\tan^{-1}\left(\sqrt{7}\right).$$

SEE ALSO:
Dirichlet Beta Function, Dirichlet Eta Function, Dirichlet Series, Double
Hecke L-Series, Modular Form, Petersson Conjecture

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