Natural operations in differential geometry

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The aim of this book is threefold:

First it should be a monographical work on natural bundles and natural operators in differential geometry. This is a field which every differential geometer has met several times, but which is not treated in detail in one place. Let us explain a little, what we mean by naturality.

Exterior derivative commutes with the pullback of differential forms. In the background of this statement are the following general concepts. The vector bundle $\La^kT^*M$ is in fact the value of a functor, which associates a bundle over M to each manifold M and a vector bundle homomorphism over f to each local diffeomorphism f between manifolds of the same dimension. This is a simple example of the concept of a natural bundle. The fact that the exterior derivative d transforms sections of $\La^kT^*M$ into sections of $\La^{k+1}T^*M$ for every manifold M can be expressed by saying that d is an operator from $\La^kT^*M$ into $\La^{k+1}T^*M$. That the exterior derivative d commutes with local diffeomorphisms now means, that d is a natural operator from the functor $\La^kT^*$ into functor $\La^{k+1}T^*$. If $k>0$, one can show that d is the unique natural operator between these two natural bundles up to a constant. So even linearity is a consequence of naturality. This result is archetypical for the field we are discussing here. A systematic treatment of naturality in differential geometry requires to describe all natural bundles, and this is also one of the undertakings of this book.

Second this book tries to be a rather comprehensive textbook on all basic structures from the theory of jets which appear in different branches of differential geometry. Even though Ehresmann in his original papers from 1951 underlined the conceptual meaning of the notion of an r-jet for differential geometry, jets have been mostly used as a purely technical tool in certain problems in the theory of systems of partial differential equations, in singularity theory, in variational calculus and in higher order mechanics. But the theory of natural bundles and natural operators clarifies once again that jets are one of the fundamental concepts in differential geometry, so that a thorough treatment of their basic properties plays an important role in this book. We also demonstrate that the central concepts from the theory of connections can very conveniently be formulated in terms of jets, and that this formulation gives a very clear and geometric picture of their properties.

This book also intends to serve as a self-contained introduction to the theory of Weil bundles. These were introduced under the name `les espaces des points proches' by A. Weil in 1953 and the interest in them has been renewed by the recent description of all product preserving functors on manifolds in terms of products of Weil bundles. And it seems that this technique can lead to further interesting results as well.

Third in the beginning of this book we try to give an introduction to the fundamentals of differential geometry (manifolds, flows, Lie groups, differential forms, bundles and connections) which stresses naturality and functoriality from the beginning and is as coordinate free as possible. Here we present the Frölicher-Nijenhuis bracket (a natural extension of the Lie bracket from vector fields to vector valued differential forms) as one of the basic structures of differential geometry, and we base nearly all treatment of curvature and Bianchi identities on it. This allows us to present the concept of a connection first on general fiber bundles (without structure group), with curvature, parallel transport and Bianchi identity, and only then add G-equivariance as a further property for principal fiber bundles. We think, that in this way the underlying geometric ideas are more easily understood by the novice than in the traditional approach, where too much structure at the same time is rather confusing. This approach was tested in lecture courses in Brno and Vienna with success.

A specific feature of the book is that the authors are interested in general points of view towards different structures in differential geometry. The modern development of global differential geometry clarified that differential geometric objects form fiber bundles over manifolds as a rule. Nijenhuis revisited the classical theory of geometric objects from this point of view. Each type of geometric objects can be interpreted as a rule F transforming every m-dimensional manifold M into a fibered manifold $FM\> M$ over M and every local diffeomorphism $f:M\>N$ into a fibered manifold morphism $Ff:FM\to FN$ over f. The geometric character of F is then expressed by the functoriality condition $F(g\o f)=Fg\o Ff$. Hence the classical bundles of geometric objects are now studied in the form of the so called lifting functors or (which is the same) natural bundles on the category $\Mf_m$ of all m-dimensional manifolds and their local diffeomorphisms. An important result by Palais and Terng, completed by Epstein and Thurston, reads that every lifting functor has finite order. This gives a full description of all natural bundles as the fiber bundles associated with the r-th order frame bundles, which is useful in many problems. However in several cases it is not sufficient to study the bundle functors defined on the category $\Mf_m$. For example, if we have a Lie group G, its multiplication is a smooth map $\mu:G\x G\to G$. To construct an induced map $F\mu:F(G\x G)\to FG$, we need a functor F defined on the whole category $\Mf$ of all manifolds and all smooth maps. In particular, if F preserves products, then it is easy to see that $F\mu$ endows FG with the structure of a Lie group. A fundamental result in the theory of the bundle functors on $\Mf$ is the complete description of all product preserving functors in terms of the Weil bundles. This was deduced by Kainz and Michor, and independently by Eck and Luciano, and it is presented in chapter VIII of this book. At several other places we then compare and contrast the properties of the product preserving bundle functors and the non-product-preserving ones, which leads us to interesting geometric results. Further, some functors of modern differential geometry are defined on the category of fibered manifolds and their local isomorphisms, the bundle of general connections being the simplest example. Last but not least we remark that Eck has recently introduced the general concepts of gauge natural bundles and gauge natural operators. Taking into account the present role of gauge theories in theoretical physics and mathematics, we devote the last chapter of the book to this subject.

If we interpret geometric objects as bundle functors defined on a suitable category over manifolds, then some geometric constructions have the role of natural transformations. Several others represent natural operators, i.e. they map sections of certain fiber bundles to sections of other ones and commute with the action of local isomorphisms. So geometric means natural in such situations. That is why we develop a rather general theory of bundle functors and natural operators in this book. The principal advantage of interpreting geometric as natural is that we obtain a well-defined concept. Then we can pose, and sometimes even solve, the problem of determining all natural operators of a prescribed type. This gives us the complete list of all possible geometric constructions of the type in question. In some cases we even discover new geometric operators in this way.

Our practical experience taught us that the most effective way how to treat natural operators is to reduce the question to a finite order problem, in which the corresponding jet spaces are finite dimensional. Since the finite order natural operators are in a simple bijection with the equivariant maps between the corresponding standard fibers, we can apply then several powerful tools from classical algebra and analysis, which can lead rather quickly to a complete solution of the problem. Such a passing to a finite order situation has been of great profit in other branches of mathematics as well. Historically, the starting point for the reduction to the jet spaces is the famous Peetre theorem saying that every linear support non-increasing operator has locally finite order. We develop an essential generalization of this technique and we present a unified approach to the finite order results for both natural bundles and natural operators in chapter V.

The primary purpose of chapter VI is to explain some general procedures, which can help us in finding all the equivariant maps, i.e. all natural operators of a given type. Nevertheless, the greater part of the geometric results is original. Chapter VII is devoted to some further examples and applications, including Gilkey's theorem that all differential forms depending naturally on Riemannian metrics and satisfying certain homogeneity conditions are in fact Pontryagin forms. This is essential in the recent heat kernel proofs of the Atiyah Singer Index theorem. We also characterize the Chern forms as the only natural forms on linear symmetric connections. In a special section we comment on the results of Kirillov and his colleagues who investigated multilinear natural operators with the help of representation theory of infinite dimensional Lie algebras of vector fields. In chapter X we study systematically the natural operators on vector fields and connections. Chapter XI is devoted to a general theory of Lie derivatives, in which the geometric approach clarifies, among other things, the relations to natural operators.
